JOURNAL OF APPROXIMATION THEORY 31, 138–153 (1981)

Condensation of Singularities and Divergence Results in Approximation Theory

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Communicated by P. L. Butzer

Received September 24, 1979

Given a family $\mathscr{A}$ of linear continuous mappings of a topological vector space X into another topological vector space Y, the set $S_{\mathscr{A}}$ of singularities for $\mathscr{A}$ is defined as the set of all x in X for which $\{A(x): A \in \mathscr{A}\}$ is an unbounded set in Y. The following general principle of condensation of singularities for nonequicontinuous families $\mathscr{A}$ is obtained: If either X is Hausdorff barrelled and Y is seminormable, or X is Hausdorff ultrabarrelled and Y is locally bounded, then $S_{\mathscr{A}}$ is an uncountably infinite dense G_δ-set in X. A principle of double condensation of singularities in Banach spaces is also obtained. These principles are applied to prove the dense unbounded divergence of Fourier series, biorthogonal systems, Lagrange interpolation processes and some quadrature formulas.

1. INTRODUCTION

It is well known that many important approximation methods such as Fourier series, Lagrange interpolation processes, some quadrature formulas, etc., are unboundedly divergent for some continuous functions called "singular functions" for the method considered. The existence of such functions is usually derived from the Banach–Steinhaus boundedness principle. But the latter tells us nothing about the cardinality and density properties of singular functions in the space of continuous functions. It is the purpose of this paper to describe the topological structure of the set of all singularities for unboundedly divergent approximation methods.

With a linear approximation method we associate an appropriate family $\mathscr{A}$ of linear continuous mappings of a topological vector space X into another topological vector space Y. The occurrence of an unbounded divergence phenomenon translates into the nonemptiness of the set of singularities for $\mathscr{A}$, i.e., the set $S_{\mathscr{A}}$ of all x in X for which $\{A(x): A \in \mathscr{A}\}$ is an unbounded set in Y. Banach and Steinhaus [1] proved the following remarkable result, called by them the principle of condensation of

0021-9045/81/020138-16$02.00/0

singularities: If X and Y are Banach spaces and $\mathscr{A}$ is not uniformly bounded, then the set $S_{\mathscr{A}}$ is dense in X and its complement in X is meagre (i.e., of first category) in X. When X is a Baire space, an extension of this principle has been obtained by Bourbaki [3].

In Section 2 we recall some needed notions and results. In Sections 3 and 4 we prove and comment the following general principle of condensation of singularities for nonequicontinuous families $\mathscr{A}$: If either X is Hausdorff barrelled and Y is seminormable, or X is Hausdorff ultrabarrelled and Y is locally bounded, then $S_{\mathscr{A}}$ is an uncountably infinite dense G_δ-set in X. The unbounded divergence phenomenon for Fourier series emphasized by Rudin [14] both in the space of functions and in the interval of their definition leads us to a principle of double condensation of singularities for one-parameter families of continuous (not necessarily linear) mappings between Banach spaces (Section 5). In Sections 6–9 we apply these principles to prove the dense unbounded divergence of Fourier series, biorthogonal systems, Lagrange interpolation processes and some quadrature formulas.

2. Preliminaries

We recall here some needed notions and results. Let T be a topological space and S a subset of T. We say that S is a *G_δ-set*, a *$G_{\delta\sigma}$-set*, or a *meagre set* in T if S can be written as an intersection of a countable family of open sets in T, as a union of a countable family of G_δ-sets in T, or as a union of a countable family of nowhere dense sets in T, respectively. We say that S is *superdense* in T if S is an uncountably infinite dense and G_δ-set in T. When each nonvoid open subset of T is nonmeagre in T, then T is said to be a *Baire space.*

Denote by K either the field R of real numbers or the field C of complex numbers, endowed with the usual topology. Let X be a topological vector space (TVS for short) over K. (Notice that X is a Baire space if and only if X is a nonmeagre set in X.) A subset M of X is said to be *bounded* if for each neighbourhood V of 0 (the origin of X) there exists a scalar $\lambda > 0$ such that $M \subset \lambda V$. A TVS X is said to be *locally bounded* if there is a bounded neighbourhood of 0 in X. A locally convex space (LCS for short) X over K is said to be a *barrelled space* if each absorbing absolutely convex and closed subset of X is a neighbourhood of 0 in X. Any Baire LCS, hence any complete semimetrizable LCS, is a barrelled space. Notice that there exist noncomplete normed spaces that are Baire spaces (cf. [6, Exercise 6.23]). The following useful characterization of barrelled spaces [13, Theorem 4] leads to the definition of a class of TVS (not necessarily LCS) having similar properties with barrelled ones.

2.1. PROPOSITION. *Let X be a* LCS *with topology $\mathscr{E}$. Then X is barrelled if and only if the only locally convex topologies on X, with bases of $\mathscr{E}$-closed neighbourhoods of* 0, *are those weaker than $\mathscr{E}$.*

A TVS X with topology $\mathscr{E}$ is said to be an *ultrabarrelled space* if the only linear topologies on X, in which there exists a base of $\mathscr{E}$-closed neighbourhoods of 0, are those weaker than $\mathscr{E}$. It follows immediately that an ultrabarrelled LCS is barrelled, but the converse may fail. Any Baire TVS, hence any complete semimetrizable TVS, is an ultrabarrelled space.

Now, denote by X and Y two TVS over the same K, and by $\mathscr{V}_X(0)$ and $\mathscr{V}_Y(0)$ the sets of neighbourhoods of 0 in X and in Y, respectively. Let $L(X, Y)$ be the vector space over K of all linear continuous mappings of X into Y. If $\mathscr{M}$ is a family of bounded subsets of X, directed by inclusion $\subset$, and $\mathscr{B}_{\mathscr{M}}$ is the family of all sets of the form

$$B(M, V) = \{A \in L(X, Y): A(M) \subset V\}, \qquad \text{with} \quad M \in \mathscr{M}, \ V \in \mathscr{V}_Y(0),$$

then there exists a unique linear topology on $L(X, Y)$ for which $\mathscr{B}_{\mathscr{M}}$ becomes a neighbourhood base of 0 in TVS $L(X, Y)$. This is called the *$\mathscr{M}$-topology* of $L(X, Y)$. When $\mathscr{M}$ consists either of all bounded or of all finite subsets of X, then the $\mathscr{M}$-topology of $L(X, Y)$ is called *uniform topology* and *pointwise topology*, respectively. A subset of $L(X, Y)$ is said to be *$\mathscr{M}$-bounded* if it is bounded in the $\mathscr{M}$-topology of $L(X, Y)$. A subset $\mathscr{A}$ of $L(X, Y)$ is said to be *equicontinuous* in $L(X, Y)$ if for each V in $\mathscr{V}_Y(0)$ there exists a U in $\mathscr{V}_X(0)$ such that $A \in \mathscr{A}$ entails $A(U) \subset V$. Any equicontinuous subset of $L(X, Y)$ is $\mathscr{M}$-bounded with respect to every $\mathscr{M}$-topology of $L(X, Y)$.

If $\mathscr{A}$ is a family of continuous mappings of a TVS X into a TVS Y, we define the *set of singularities* for $\mathscr{A}$ as the set $S_{\mathscr{A}}$ of all points x in X for which $\{A(x): A \in \mathscr{A}\}$ is an unbounded set in Y.

3. CONDENSATION OF SINGULARITIES IN BARRELLED AND ULTRABARRELLED SPACES

The following theorem describes the topological structure of the set of singularities for nonequicontinuous families of linear and continuous mappings.

3.1. THEOREM. *Let X and Y be two* TVS *over the same K, and let $\mathscr{A}$ be a subset of $L(X, Y)$ which is not equicontinuous in $L(X, Y)$.*

(i) *Then $S_{\mathscr{A}}$ contains the intersection of a countable family of open and dense sets in X.*

(ii) *If either X is ultrabarrelled, or X is barrelled and Y is a* LCS, *then $S_{\mathscr{A}}$ contains a dense G_δ-set in X; if, in addition, X is Hausdorff, then $S_{\mathscr{A}}$ contains a superdense set in X.*

(iii) *If either X is ultrabarrelled and Y is semimetrizable, or X is barrelled and Y is a semimetrizable* LCS, *then $S_{\mathscr{A}}$ is a dense $G_{\delta\sigma}$-set in X.*

(iv) *If either X is Hausdorff ultrabarrelled and Y is locally bounded, or X is Hausdorff barrelled and Y is seminormable, then $S_{\mathscr{A}}$ is superdense in X.*

Proof. Let V be a neighbourhood of 0 in Y with the help of which we write that $\mathscr{A}$ is not equicontinuous. For each W in $\mathscr{V}_Y(0)$ we define the set $S_{\mathscr{A},W}$ by

$$S_{\mathscr{A},W} = \bigcap \{X_{n,W}: n \in N\}, \tag{3.1}$$

where

$$X_{n,W} = \bigcup \{\{x \in X: A(x) \notin nW\}: A \in \mathscr{A}\}. \tag{3.2}$$

(i) Let W be a closed balanced neighbourhood of 0 in Y such that $W + W \subset V$. The sets $X_{n,W}$ in (3.2) are open in X for all $n \in N$ because the sets $\{x \in X: A(x) \notin nW\} = A^{-1}(Y \backslash nW)$ are open for all $A \in \mathscr{A}$. Since $\bigcap \{X_{n,W}: n \in N\} \subset S_{\mathscr{A}}$, it remains to prove that each $X_{n,W}$ is dense in X. Supposing the contrary, there exist $n_0 \in N$, $x_0 \in X$ and $U_0 \in \mathscr{V}_X(0)$ such that $(x_0 + U_0) \cap X_{n_0,W} = \varnothing$. Then

$$A(x) = A(x_0 + x) - A(x_0) \in n_0 W - n_0 W = n_0(W + W) \subset n_0 V$$

for all $x \in U_0$ and all $A \in \mathscr{A}$, which contradicts the choice of V. Hence, the family $\{X_{n,W}: n \in N\}$ fulfills the requirements in (i).

(ii) First, we assume that X is ultrabarrelled. The family

$$\mathscr{W} = \{W \in \mathscr{V}_Y(0): W \text{ is closed balanced and } W \subset V\}$$

is a neighbourhood base of 0 in Y. The sets $S_{\mathscr{A},W}$, $W \in \mathscr{W}$, defined in (3.1), are G_δ-sets and satisfy $S_{\mathscr{A},W} \subset S_{\mathscr{A}}$. To prove the first affirmation in (ii), it suffices to show that there exists a W in $\mathscr{W}$ such that $S_{\mathscr{A},W}$ be a dense set in X.

Suppose the contrary. Then the set family

$$\mathscr{U} = \{X_W: W \in \mathscr{W}\}, \qquad \text{where} \quad X_W = \bigcap \{A^{-1}(W): A \in \mathscr{A}\},$$

has the properties:

(a) each X_W is a balanced and absorbing set in X,

(b) if X_{W_1} and X_{W_2} are in $\mathscr{U}$, there exists an X_{W_3} in $\mathscr{U}$ such that $X_{W_3} \subset X_{W_1} \cap X_{W_2}$,

(c) if X_{W_1} is in $\mathscr{U}$, there exists an X_{W_2} in $\mathscr{U}$ with $X_{W_2} + X_{W_2} \subset X_{W_1}$.

Indeed, it is clear that X_W, $W \in \mathscr{W}$, is balanced. Next, let $W_1 \in \mathscr{V}_Y(0)$ be a closed balanced set with $W_1 + W_1 \subset W$. Since the set $S_{\mathscr{A}, W_1}$ is not dense in X (note that $W_1 \in \mathscr{W}$), there exist $x_0 \in X$ and $U_0 \in \mathscr{V}_X(0)$ such that $(x_0 + U_0) \cap S_{\mathscr{A}, W_1} = \varnothing$. To prove that X_W is an absorbing set in X, let x be in X. Choosing a $\lambda > 0$ with $\lambda x \in U_0$, the elements x_0 and $x_0 + \lambda x$ are not in $S_{\mathscr{A}, W_1}$, hence there is an $n_0 \in N$ so that x_0 and $x_0 + \lambda x$ are not in X_{n_0, W_1}. Now, by definition (3.2) of X_{n_0, W_1}, we have

$$A(n_0^{-1} \lambda x) = A(n_0^{-1}(x_0 + \lambda x)) - A(n_0^{-1} x_0) \in W_1 + W_1 \subset W$$

for all $A \in \mathscr{A}$. Therefore, $n_0^{-1} \lambda x \in \bigcap \{A^{-1}(W) : A \in \mathscr{A}\} = X_W$ and property (a) holds.

Properties (b) and (c) hold with a $W_3 \in \mathscr{W}$ such that $W_3 \subset W_1 \cap W_2$, and with a $W_2 \in \mathscr{W}$ such that $W_2 + W_2 \subset W_1$, respectively.

Thus, by a well-known result (cf. [3, Chap. I, Sects. 1, 5, Remarque 2]), there exists a unique linear topology $\mathscr{T}'$ on X for which $\mathscr{U}$ becomes a neighbourhood base of 0 in X. Since every X_W in $\mathscr{U}$ is a closed set in the initial topology $\mathscr{T}$ on X, and X is ultrabarrelled, the topology $\mathscr{T}'$ is weaker than $\mathscr{T}$, so that $\bigcap \{A^{-1}(W) : A \in \mathscr{A}\} = X_W$ is in $\mathscr{V}_X(0)$ for all $W \in \mathscr{W}$. Hence, $A(X_W) \subset W \subset V$ for a W in $\mathscr{V}_Y(0)$ and all A in $\mathscr{A}$, which contradicts the choice of V.

When X is barrelled and Y is a LCS, the preceding argument holds if the neighbourhood base $\mathscr{W}$ is replaced by

$$\{W \in \mathscr{V}_Y(0) : W \text{ is closed absolutely convex and } W \subset V\},$$

Proposition 2.1 is invoked, and the property (a) for $\mathscr{U}$ is replaced by

(a′) each X_W is absolutely convex and absorbing set in X.

If, in addition, X is Hausdorff then, by the choosing of V, there exists an x in $S_{\mathscr{A}, W}$ with $x \neq 0$, where W is a neighbourhood of 0 in Y such that $S_{\mathscr{A}, W}$ is dense in X. It follows immediately that λx is in $S_{\mathscr{A}, W}$ for all $\lambda > 0$, which shows that the set $S_{\mathscr{A}, W}$ is uncountably infinite. Hence, $S_{\mathscr{A}, W} \subset S_{\mathscr{A}}$ and $S_{\mathscr{A}, W}$ is a superdense set in X.

(iii) Suppose that either X is ultrabarrelled and Y is semimetrizable, or X is barrelled and Y is a semimetrizable LCS. Then there exists a countable base of closed balanced neighbourhoods $W_i \subset V$, $i \in N$, of 0 in Y. If $x \in S_{\mathscr{A}}$, there exists an $i \in N$ such that for each $n \in N$ one can find an $A \in \mathscr{A}$ with $A(x) \notin n W_i$, hence $x \in S_{\mathscr{A}, W_i}$, where $S_{\mathscr{A}, W_i}$ is defined by (3.1) and (3.2).

Conversely, the argument in (ii) shows that $S_{\mathscr{A}} = \bigcup \{S_{\mathscr{A},W_i}: i \in N\}$ is a dense $G_{\delta\sigma}$-set in X.

(iv) Finally, suppose that either X is Hausdorff ultrabarrelled and Y is locally bounded, or X is Hausdorff barrelled and Y is seminormable. Then there exists a closed balanced bounded neighbourhood V_0 of 0 in Y. We shall show that $S_{\mathscr{A}} = S_{\mathscr{A},V_0}$, where $S_{\mathscr{A},V_0}$ is defined by (3.1) and (3.2). To this end, let x be in $S_{\mathscr{A}}$. Since the set $\{A(x): A \in \mathscr{A}\}$ is not bounded in Y, there exists a closed balanced neighbourhood W of 0 in Y such that for each $\alpha > 0$ there is an A in $\mathscr{A}$ with $A(x) \notin \alpha W$. For the bounded set V_0 we can take a $\lambda > 0$ with $V_0 \subset \lambda W$. By the choosing of W, for each $n \in N$ there exists an $A \in \mathscr{A}$ with $A(x) \notin n\lambda W$, hence $A(x) \notin nV_0$. This implies $x \in X_{n,V_0}$ for all $n \in N$, whence $x \in S_{\mathscr{A},V_0}$. Now, from (ii) it follows that the set $S_{\mathscr{A}} = S_{\mathscr{A},V_0}$ is superdense in X.

This completes the proof of the theorem.

4. Examples and Remarks

4.1. Example. We shall show that the set U of all unbounded sequences is superdense in the LCS s of all scalar sequences $x = (x_1, ..., x_n, ...)$, equipped with the seminorms $p_n(x) = \max\{|x_1|, ..., |x_n|\}$, $n \in N$.

To this end we introduce the vector space l^0 of all scalar sequences $x = (x_1, ..., x_n, ...)$ with at most a finite number of nonzero terms, endowed with the norm $\|x\| = \max\{|x_1|, ..., |x_n|, ...\}$. Notice that U coincides with the set $S_{\mathscr{A}}$ of singularities for the family $\mathscr{A}$ of linear continuous mappings $A_n: s \to l^0$, $n \in N$, given by

$$A_n(x) = (x_1, ..., x_n, 0, 0, ...), \qquad x = (x_1, ..., x_n, ...) \in s.$$

Since s is a complete metrizable LCS with the metric

$$\rho(x, y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|(1 + |x_n - y_n|)^{-1},$$

the above result follows from Theorem 3.1(iv), as soon as we verify that $\mathscr{A}$ is not equicontinuous in $L(s, l^0)$. For $V = \{y \in l^0: \|y\| < 1\}$ and each $W = \{x \in s: p_n(x) < r\}$, $n \in N$, $r > 0$, we have $e_{n+1} = (0, ..., 0, 1, 0, ...) \in W$ (1 is at the $(n+1)$st place), but $A_{n+1}(e_{n+1}) = e_{n+1} \notin V$, hence $\mathscr{A}$ is not equicontinuous.

4.2. *Remark.* Since the equicontinuous families in $L(X, Y)$ are bounded in the pointwise topology of $L(X, Y)$, Theorem 3.1(ii), contains the Banach–Steinhaus boundedness principle both in Bourbaki's form [3, Chap. III, Sect. 3, Théorème 2, Corollaire] and in Robertson's form [13, Theorem 5]: *If either X is barrelled and Y is a* LCS, *or X is ultrabarrelled, then every pointwisely bounded set in $L(X, Y)$ is equicontinuous in $L(X, Y)$.* On the other hand, if X is either a LCS or a TVS for which the conclusion of Banach–Steinhaus boundedness principle holds with $Y = K$ and with every TVS Y, respectively, then X is necessarily barrelled in the first case, and ultrabarrelled in the second case (cf. [6, Theorem 7.1.1(b)]; [15, pp. 10–12]). Consequently, the barrelled and ultrabarrelled spaces constitute the most general framework in which a principle of condensation of singularities as Theorem 3.1 holds. The next example is an illustration of the last remark.

4.3. EXAMPLE. The normed space l^0 in Example 4.1 is neither barrelled nor ultrabarrelled, since the closed absolutely convex absorbing set $\{(x_1,..., x_n,...) \in l^0: n\,|x_n| \leqslant 1$ for all $n \in N\}$ is not a neighbourhood of 0 in l^0. The family $\mathscr{A}$ of linear continuous functionals $A_n: l^0 \to K$, $n \in N$, given by $A_n(x) = nx_n$, is not equicontinuous in $L(l^0, K)$, although $S_{\mathscr{A}} = \varnothing$.

4.4. *Remark.* Theorem 3.1 includes the principle of condensation of singularities not only in the Banach–Steinhaus' form [1], but also in the more general Bourbaki's formulation [3, Chap. III, Sect. 3, Exercise 15]: *If X is a Baire* TVS, *Y is a* TVS *and $\mathscr{A}$ is a nonequicontinuous set in $L(X, Y)$, then the set $S_{\mathscr{A}}$ is dense in X and its complement in X is meagre in X.* Indeed, by Theorem 3.1(ii), $S_{\mathscr{A}}$ contains a set S which is dense in X and has the form $S = \bigcap \{X_n: n \in N\}$, where X_n are open sets in X; hence, the sets $Z_n = X \setminus (S_{\mathscr{A}} \cup X_n)$ satisfy $\operatorname{int} \bar{Z}_n = \varnothing$ and $X \setminus S_{\mathscr{A}} = \bigcup \{Z_n: n \in N\}$.

The following example shows that the principle of condensation of singularities in Theorem 3.1(ii), is effectively more general than that of Bourbaki.

4.5. EXAMPLE. The vector space l^0 in Example 4.1, endowed with the finest locally convex topology on l^0, becomes a Hausdorff barrelled space (see Proposition 2.1). But l^0 is not a Baire space. Indeed, we have $l^0 = \bigcup \{X_n: n \in N\}$, where the sets

$$X_n = \{(x_1,..., x_k,...) \in l^0: \text{with } x_k = 0 \text{ for all } k > n\}$$

are closed in l^0 as finitedimensional vector subspaces of l^0. Moreover, X_n are nowhere dense in l^0 since, if $\operatorname{int} X_n = \operatorname{int} \bar{X}_n \neq \varnothing$ for an $n \in N$, then X_n would be an absorbing set in l^0, so for the sequence $e_{n+1} = (0,..., 0, 1, 0,...) \in l^0$ there must exist a $\lambda > 0$ such that $\lambda e_{n+1} \in X_n$, which is a contradiction. Consequently, X is not a Baire space.

The family of linear continuous functionals $A_\alpha: l^0 \to K$, $\alpha \in K$, given by $A_\alpha(x) = \alpha x_1$, $x = (x_1,..., x_n,...) \in l^0$, is not equicontinuous in $L(l^0, K)$, so Theorem 3.1(ii), applies.

To the same purpose answers the space of indefinitely derivable functions in the theory of distributions. This is a Hausdorff barrelled space without being a Baire space.

5. Double Condensation of Singularities

In this section we establish a principle of double condensation of singularities for one-parameter families of continuous (not necessarily linear) mappings between normed spaces. First, we recall a needed lemma whose proof may be found in [14, p. 103]:

5.1. Lemma. *Let T be a nonvoid complete metric space without isolated points. Then the intersection of any countable family of open dense sets in T is superdense in T.*

5.2. Theorem. *Let X be a nonzero Banach space, Y a normed space, and T a nonvoid separable complete metric space without isolated points. Let also $\mathscr{A} = \{A_i: i \in I\}$ be a family of mappings of $X \times T$ into Y satisfying the following conditions:*

(a) *$A_i(\cdot, t): X \to Y$ is continuous, $\|A_i(x+y, t)\| \leqslant \|A_i(x, t)\| + \|A_i(y, t)\|$ and $\|A_i(\lambda x, t)\| \leqslant \|A_i(x, t)\|$ for each $i \in I$, $t \in T$, $x, y \in X$ and $\lambda \in K$ with $|\lambda| \leqslant 1$,*

(b) *$A_i(x, \cdot): T \to Y$ is continuous for each $i \in I$ and $x \in X$,*

(c) *there exists a dense set T_0 in T such that*

$$\sup\{\|A_i(x, t)\|: x \in X, \|x\| \leqslant 1, \text{ and } i \in I\} = \infty \qquad \text{for all } t \in T_0.$$

Then there exists a superdense set X_0 in X such that for every $x \in X_0$ the set $\{t \in T: \sup\{\|A_i(x, t)\|: i \in I\} = \infty\}$ is superdense in T.

Proof. Since T_0 is a dense set in the separable metric space T, there exists a countable subset $T_0' = \{t_n: n \in N\}$ of T_0 which is dense in T. Condition (a) implies that the functions $f_n: X \to [0, \infty]$, $n \in N$, given by

$$f_n(x) = \sup\{\|A_i(x, t_n)\|: i \in I\},$$

are lower semicontinuous, and $f_n(x+y) \leqslant f_n(x) + f_n(y)$ and $f_n(\lambda x) \leqslant f_n(x)$ for all $n \in N$, $x, y \in X$ and $\lambda \in K$ with $|\lambda| \leqslant 1$. Condition (c) yields the unboundedness of the functions f_n on the closed unit ball of X. Then, by a known result (cf. [6, Theorem 7.5.1]), there exists a subset S of X such that $X \backslash S$ is a meagre set in X and $f_n(x) = \infty$ for all $n \in N$ and all $x \in S$. The Baire theorem ensures that S is a dense set in X.

Now let X_0 be the set $\bigcap \{X_{n,k}: n, k \in N\}$, where $X_{n,k} = \{x \in X: f_n(x) > k\}$. Since f_n are lower semicontinuous and $S \subset X_0$, by Lemma 5.1 we conclude that X_0 is a superdense set in X. By condition (b), the sets

$$T_{m,i}(x) = \{t \in T: \|A_i(x, t)\| > m\} \qquad \text{and} \qquad T_m(x) = \bigcup \{T_{m,i}(x): i \in I\}$$

are open in T for all $x \in X$, $m \in N$ and $i \in I$.

Let x be an element in X_0. We shall show that the sets $T_m(x)$, $m \in N$, are dense in T. To this end, let t be a point in T and let V be a neighbourhood of t in T. Since T_0' is dense in T, there exists a $t_n \in T_0' \cap V$. Since $f_n(x) = \infty$, it follows that $t_n \in T_m(x)$ for all $m \in N$, hence $t_n \in T_m(x) \cap V$. Thus, the sets $T_m(x)$ are dense in T. Using once again Lemma 5.1, we conclude that the set

$$\{t \in T: \sup\{\|A_i(x, t)\|: i \in I\} = \infty\} = \bigcap \{T_m(x): m \in N\}$$

is superdense in T, and the proof is complete.

5.3. *Remark.* The next example shows that the density hypothesis in condition (c) of Theorem 5.2 cannot be dropped. Let $X = Y = R$, $I = N$, $T = [0, 2]$ and let $A_i: R \times [0, 2] \to R$ be defined by $A_i(x, t) = xt^i$. Conditions (a) and (b) in Theorem 5.2 are fulfilled. The set of all t in $[0, 2]$, for which

$$\sup\{|x|\, t^i: |x| \leqslant 1, \text{ and } i \in N\} = \infty,$$

coincides with the interval $]1, 2]$, hence the density hypothesis fails. The conclusion of Theorem 5.2 fails too, since for any superdense set X_0 in X and any x in X_0 we have $\{t \in [0, 2]: \sup\{|x|\, t^i: i \in N\} = \infty\} \subset\,]1, 2]$.

5.4. THEOREM. *Let X be a nonzero Banach space, Y a normed space and $\mathscr{A} = \{A_i: i \in I\}$ a family of continuous mappings of X into Y satisfying the following conditions*:

(a) $\|A_i(x + y)\| \leqslant \|A_i(x)\| + \|A_i(y)\|$ *and* $\|A_i(\lambda x)\| \leqslant \|A_i(x)\|$ *for each* $i \in I$, $x, y \in X$ *and* $\lambda \in K$ *with* $|\lambda| \leqslant 1$,

(b) $\sup\{\|A_i(x)\|: x \in X, \|x\| \leqslant 1, \text{ and } i \in I\} = \infty$.

Then the set $S_{\mathscr{A}} = \{x \in X: \sup\{\|A_i(x)\|: i \in I\} = \infty\}$ of singularities for $\mathscr{A}$ is superdense in X.

Proof. We use the argument in the first part of the proof of Theorem 5.2, in which the functions f_n are all given by $f_n(x) = f(x) = \sup\{\|A_i(x)\|: i \in I\}$, and the sets $X_{n,k}$ are replaced by $X_k = \{x \in X: f(x) > k\}$, $k \in N$. We obtain that the set $S_{\mathscr{A}} = \bigcap \{X_k: k \in N\}$ is superdense in X.

5.5. EXAMPLE. Let $(u_n)_{n \in N}$ be an unbounded scalar sequence, and let l^1

be the Banach space of all summable sequences $x=(x_n)_{n\in N}$, endowed with the norm $\|x\|=\sum_{n=1}^{\infty}|x_n|$. We shall show that the set of all sequences (x_n) in l^1, for which $\sum_{n=1}^{\infty}|x_n u_n|=\infty$, is superdense in l^1.

To this end we use Theorem 5.4 with $X=l^1$, $Y=R$ and $\mathscr{A}=\{A_n: n\in N\}$, where the continuous (nonlinear) functionals $A_n: l^1\to R$ are given by

$$A_n(x)=\sum_{k=1}^{n}|x_k u_k|, \qquad x=(x_k)\in l^1.$$

The condition (a) is obviously fulfilled. The condition (b) is implied by

$$\sup\{|A_n(x)|: x\in l^1, \|x\|\leqslant 1\}\geqslant |A_n(x_n^*)|=|u_n|,$$

where $x_n^*=(0,..., 0, \operatorname{sign} u_n, 0,...)$. Hence, the set $S_{\mathscr{A}}=\{(x_1,..., x_k,...)\in l^1: \sum_{k=1}^{\infty}|x_k u_k|=\infty\}$ is superdense in l^1.

6. Divergence of Fourier Series

Let T be the interval $[0, 1]$ and $e_k: T\to C$, $k\in Z$, be the functions defined by $e_k(t)=\exp(2\pi ikt)$. A classical theorem asserts that for each continuous (even measurable with integrable square) function $x: T\to C$ the associated Fourier series

$$\sum_{k\in Z} c_k e_k, \qquad \text{where} \quad c_k=(x\mid e_k), \tag{6.1}$$

converges to x in the Hilbert space $L^2(T)$. The problem of pointwise convergence of this series to x was solved in the negative by du Bois-Reymond (1876), who, for each t in T, exhibited a continuous function on T having its Fourier series divergent at t. Given an $x\in C(T)$, denote by $UD(x)$ the set of all $t\in T$ at which the Fourier series of x is unboundedly divergent. Bari [2, pp. 318–320] constructed a function $x\in C(T)$ with the property that $UD(x)$ is a superdense set in T. Rudin [14, pp. 101–103] showed that the set of all functions x in $C(T)$ having the last property contains a superdense set in $C(T)$. We derive Rudin's result as an application of Theorem 5.2.

Given $x\in C(T)$, $t\in T$ and $n\in N$, we consider the partial sum of series (6.1):

$$A_n(x, t)=\sum_{k=-n}^{n} c_k e_k(t). \tag{6.2}$$

6.1. Theorem (Rudin [14]). *There exists a superdense set X_0 in $C(T)$ such that for each $x\in X_0$ the set $UD(x)=\{t\in T: \sup\{|A_n(x, t)|: n\in N\}=\infty\}$ is superdense in T.*

Proof. In Theorem 5.2 take for X the complex Banach space $C(T)$ with respect to uniform norm, and for $\mathscr{A}$ the family of linear continuous functionals $A_n(\cdot, t)$: $C(T) \to C$, $n \in N$, $t \in T$, defined by (6.2). Since

$$\begin{aligned}
\|A_n(\cdot, t)\| &= \int_T \left| \frac{\sin(2n+1)(t-s)\pi}{\sin(t-s)\pi} \right| ds \\
&= \int_T \left| \frac{\sin(2n+1)s\pi}{\sin s\pi} \right| ds > \int_T \frac{|\sin(2n+1)s\pi|}{s\pi} ds \\
&> \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{2k\pi} \int_{(2k-1)\pi}^{2k\pi} |\sin s| \, ds \\
&= \frac{2}{\pi^2} \sum_{k=1}^{n} \frac{1}{k} \to \infty \qquad \text{as} \quad n \to \infty,
\end{aligned}$$

the condition (c) in Theorem 5.2 is fulfilled with $T_0 = T$.

6.2. *Remark.* Theorem 6.1 contrasts with Carleson's famous result (1966): for each x in $L^2(T)$ the series (6.1) converges to x almost everywhere on T.

7. Divergence of Biorthogonal Systems

In this section we establish a variant of Theorem 6.1 for general biorthogonal systems in topological vector spaces [5].

Let X be a TVS and X^* its topological dual space. A *biorthogonal system* in X is a sequence $((x_i, f_i))_{i \in N}$ in $X \times X^*$ such that $f_i(x_j) = \delta_{i,j}$, $i, j \in N$. A biorthogonal system $((x_i, f_i))$ in X is said to be: (a) *X-complete* if the vector space spanned by $\{x_1, x_2, \ldots\}$ is dense in X, and (b) a *Schauder basis* for X if for each $x \in X$ we have

$$x = \sum_{i=1}^{\infty} f_i(x) \, x_i,$$

the series being convergent in the topology of X. The partial sum operators $s_n: X \to X$, $n \in N$, associated with $((x_i, f_i))$, are given by

$$s_n(x) = \sum_{i=1}^{n} f_i(x) \, x_i, \qquad x \in X.$$

It is clear that every Schauder basis for X is an X-complete biorthogonal system in X. The converse is not more true. Of course, using the notations and results in Section 6, we see that the sequence $((e_k, f_k))_{k \in Z}$, where

$$f_k(x) = \int_T x(t) \, \overline{e_k(t)} \, dt, \qquad x \in C(T),$$

is a $C(T)$-complete biorthogonal system in $C(T)$ which is not a Schauder basis for $C(T)$. Moreover, Theorem 5.4 entails that for each $t \in T$ the set of singularities $\{x \in C(T): \sup\{|A_n(x, t)|: n \in N\} = \infty\}$ is superdense in $C(T)$. The following theorem is a similar result for general biorthogonal systems.

7.1. THEOREM. *Let X be a Hausdorff* TVS. *Suppose that either X is ultrabarrelled, or X is barrelled. If $((x_i, f_i))_{i\in N}$ is an X-complete biorthogonal system in X which is not a Schauder basis for X, then the set of all x in X such that the set $\{s_n: n \in N\}$ is unbounded in X contains a superdense set in X.*

The proof depends on the following immediate extension of a known criterion of pointwise convergence in Banach spaces:

7.2. PROPOSITION. *Let X and Y be two* TVS *over the same K and let $A_n \in L(X, Y)$, $n = 0, 1,\dots$. In order that $A_n(x) \to A_0(x)$ for all $x \in X$ as $n \to \infty$ it is sufficient that the following two conditions be fulfilled:*

(a) *there exists a dense subset X' of X such that $A_n(x') \to A_0(x')$ for all $x' \in X'$ as $n \to \infty$,*

(b) *the set $\{A_n: n \geq 0\}$ is equicontinuous in $L(X, Y)$.*

Conversely, if either X is ultrabarrelled or X is barrelled and Y is a LCS, *then the conditions* (a) *and* (b) *are also necessary for $A_n(x) \to A_0(x)$ for all $x \in X$ as $n \to \infty$.*

Proof. Suppose that the conditions (a) and (b) are fulfilled. Let $x \in X$ and $V \in \mathscr{V}_Y(0)$. Choose a $W \in \mathscr{V}_Y(0)$ such that $W + W + W \subset V$. By the equicontinuity of $\{A_n: n \geq 0\}$, there exists a balanced neighbourhood $U \in \mathscr{V}_X(0)$ such that $A_n(U) \subset W$ for all $n \geq 0$. Since the set X' is dense in X, there is a $x' \in X'$ with $x - x' \in U$. Now, $A_n(x') \to A_0(x')$ as $n \to \infty$ ensures the existence of an $n_0 \in N$ with $A_n(x') - A_0(x') \in W$ for all $n \geq n_0$. Thus,

$$\begin{aligned} A_n(x) - A_0(x) &= A_n(x - x') + A_n(x') - A_0(x') \\ &\quad + A_0(x' - x) \in A_n(U) + W + W \subset V, \end{aligned}$$

hence $A_n(x) \to A_0(x)$ as $n \to \infty$.

Let us suppose conversely, that either X is ultrabarrelled or X is barrelled and Y is a LCS, and that $A_n(x) \to A_0(x)$ for all $x \in X$ as $n \to \infty$. The condition (a) is trivially satisfied with $X' = X$. Since the sequence $(A_n)_{n\in N}$ is pointwisely convergent, the set $\{A_n: n \geq 0\}$ is pointwisely bounded in $L(X, Y)$, hence, by Remark 4.2, the set $\{A_n: n \geq 0\}$ is equicontinuous in $L(X, Y)$.

Proof of Theorem 7.1. Apply the sufficiency part of Proposition 7.2 with $X' = \text{span}\{x_1, x_2, \ldots\}$, $A_n = s_n$ and $A_0 =$ the identity operator in X. Then, since $((x_i, f_i))$ is X-complete but it is not a Schauder basis for X, it follows that the set $\{s_n : n \in N\}$ is not equicontinuous in $L(X, X)$. Now, the conclusion of Theorem 7.1 is a consequence of Theorem 3.1(ii).

8. Divergence of Lagrange Interpolation Processes

Let M be a triangular matrix of distinct nodes $t_n^1 < \cdots < t_n^n$, $n \in N$, in the interval $T = [-1, 1]$. Given a function x in $C(T)$, denote by $L_n(x, \cdot)$ the Lagrange interpolation polynomial of x over the nodes $t_n^1, \ldots, t_n^n$ in M, defined by

$$L_n(x, t) = \sum_{k=1}^{n} x(t_n^k)\, l_n^k(t), \qquad t \in T, \tag{8.1}$$

where

$$l_n^k(t) = \omega_n(t)/[(t - t_n^k)\, \omega_n'(t_n^k)] \qquad \text{and} \qquad \omega_n(t) = (t - t_n^1) \cdots (t - t_n^n).$$

The problem of convergence of the sequence $(L_n(x, \cdot))_{n \in N}$ to x in various senses has been preoccupying many mathematicians of our century. The first was Runge (1901), who for equidistant nodes exhibited an analytic function for which the sequence of Lagrange interpolation polynomials diverges on some intervals. For equidistant nodes with $t_n^1 = -1$ and $t_n^n = 1$, and for the function $x(t) = |t|$, $t \in T$, Bernstein (1916) showed that

$$\sup\{|L_n(x, t)| : n \in N\} = \infty \tag{8.2}$$

on the whole interval $T = [-1, 1]$ except the points $-1, 0, 1$. In the case of arbitrary node matrices, Faber (1914) proved the existence of a function x in $C(T)$ for which the sequence $(L_n(x, \cdot))_{n \in N}$ does not converge uniformly to x on the interval T. Moreover, one of us [9] has showed that for each node matrix the set $\{x \in C(T) : \sup\{\|L_n(x, \cdot)\|_{C(T)} : n \in N\} = \infty\}$ is superdense in $C(T)$.

Theorem 8.1 emphasizes the phenomenon of double condensation of singularities for Lagrange interpolation processes. Its proof is based on the following deep result of Erdös [7]: for each node matrix M there exists a subset E of T with mes $E = 2$ such that

$$\overline{\lim_{n \to \infty}} \sum_{k=1}^{n} |l_n^k(t)| = \infty \qquad \text{for all } t \in E. \tag{8.3}$$

8.1. THEOREM. *Given a node matrix M, there exists a superdense set X_0 in $C(T)$ such that for each x in X_0 the set $\{t \in T\colon \sup\{|L_n(x,t)|\colon n \in N\} = \infty\}$ is superdense in T.*

Proof. The functionals $L_n(\cdot, t)\colon C(T) \to R$, $t \in T$, defined by (8.1), are linear and continuous and their norm is given by

$$\|L_n(\cdot, t)\| = \sum_{k=1}^{n} |l_n^k(t)|, \qquad n \in N,\ t \in T.$$

Now, taking into account of (8.3), Theorem 5.2 applies with $X = C(T)$, $Y = R$, $I = N$, $T = [-1, 1]$, $A_n(x, t) = L_n(x, t)$ and $T_0 = E$.

8.2. *Remarks.* Pilipčuk [10] has proved that, given an arbitrary node matrix M, there exists a subset E of T with $\operatorname{mes} E = 2$ such that for each $t \in E$ one can find a function x in $C(T)$ satisfying both (8.2) and the supplementary conditions $x(t) = 0$ and $\int_T |x(s)/(s-t)|\, ds < \infty$. When the nodes in M are the roots of Jacobi polynomials $P_n^{(\alpha,\beta)}$ with $\min\{\alpha, \beta\} > -1$, Privalov [12] has showed that there exists a function x in $C(T)$ satisfying (8.2) almost everywhere on T, and recently Pilipčuk [11] has exhibited a function x in $C(T)$ with preassigned modulus of continuity such that the set $\{t \in T\colon \sup\{|L_n(x,t)|\colon n \in N\} = \infty\}$ is superdense in T. Moreover, in the case of Čebyšev nodes ($\alpha = \beta = -1/2$) Grünwald (1936) and Marcinkiewicz (1937) constructed a continuous function for which (8.2) holds everywhere on T.

In contrast with the preceding divergence results, Erdös and Turán (1937) proved that for each node matrix M and each x in $C(T)$ the sequence $(L_n(x, \cdot))_{n \in N}$ converges to x in the Hilbert space $L^2(T)$.

9. DIVERGENCE OF SOME QUADRATURE FORMULAS

Let $(m_n)_{n \in N}$ be a sequence of natural numbers, $c_n^0, c_n^1, \ldots, c_n^{m_n}$ a matrix of real coefficients, and $-1 \leqslant t_n^0 < t_n^1 < \cdots < t_n^{m_n} \leqslant 1$ a matrix of nodes in the interval $T = [-1, 1]$. We present some conditions on the coefficients or on the nodes which entail the unbounded divergence of quadrature formulas

$$\int_T x(t)\, dt = Q_n(x) + R_n(x), \qquad x \in C(T), \tag{9.1}$$

where

$$Q_n(x) = \sum_{k=0}^{m_n} c_n^k x(t_n^k). \tag{9.2}$$

9.1. THEOREM. *Suppose that one of the following conditions is fulfilled*:

$$(c_1) \qquad \sup\left\{\sum_{k=0}^{m_n} |c_n^k| : n \in N\right\} = \infty$$

or

(c_2) *$m_n = n$, the nodes t_n^k, $k = 0, 1, \ldots, n$, are the roots of Jacobi polynomial $P_{n+1}^{(\alpha,\beta)}$ with* $\max\{\alpha, \beta\} > 3/2$, *and $R_n(p) = 0$ for any polynomial p of degree $\leqslant n$. Then the set $\{x \in C(T): \sup\{|Q_n(x)|: n \in N\} = \infty\}$ is superdense in $C(T)$.*

Proof. It is easy to see that the functionals $Q_n: C(T) \to R$ defined by (9.2) are linear and continuous on the real Banach space $C(T)$, and that their norm is given by

$$\|Q_n\| = \sum_{k=0}^{m_n} |c_n^k|, \qquad n \in N. \tag{9.3}$$

Theorem 3.1(iv) (or Theorem 5.4) applies whenever we are convinced of

$$\sup\{\|Q_n\|: n \in N\} = \infty. \tag{9.4}$$

If condition (c_1) is fulfilled, then (9.3) implies (9.4). If condition (c_2) holds, Locher [8] proved that there exists a constant $c > 0$ such that

$$\|R_n\| = 2 + \sum_{k=0}^{n} |c_n^k| \geqslant c \cdot n^{\max\{\alpha,\beta\} - 3/2}$$

for all sufficiently large $n \in N$. Hence $\|Q_n\| \geqslant -2 + c \cdot n^{\max\{\alpha,\beta\} - 3/2}$ and so (9.4) is satisfied.

9.2. *Remark.* Suppose that Q_n in (9.1) is given by

$$Q_n(x) = \int_T L_n(x, t)\, dt, \qquad x \in C(T),$$

where $L_n(x, \cdot)$ is the Lagrange interpolation polynomial of x over the equidistant nodes $t_n^k = -1 + k/n$, $k = 0, 1, \ldots, 2n$. Brass [4] showed that $(-1)^n Q_n(x) \to \infty$ as $n \to \infty$ for the function $x(t) = |t|$, $t \in T$, which immediately entails condition (c_1) of Theorem 9.1. Therefore, the set of all functions x in $C(T)$, for which the Newton–Côtes quadrature formula

$$\int_T x(t)\, dt = \int_T L_n(x, t)\, dt + R_n(x)$$

diverges unboundedly, is superdense in $C(T)$.

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